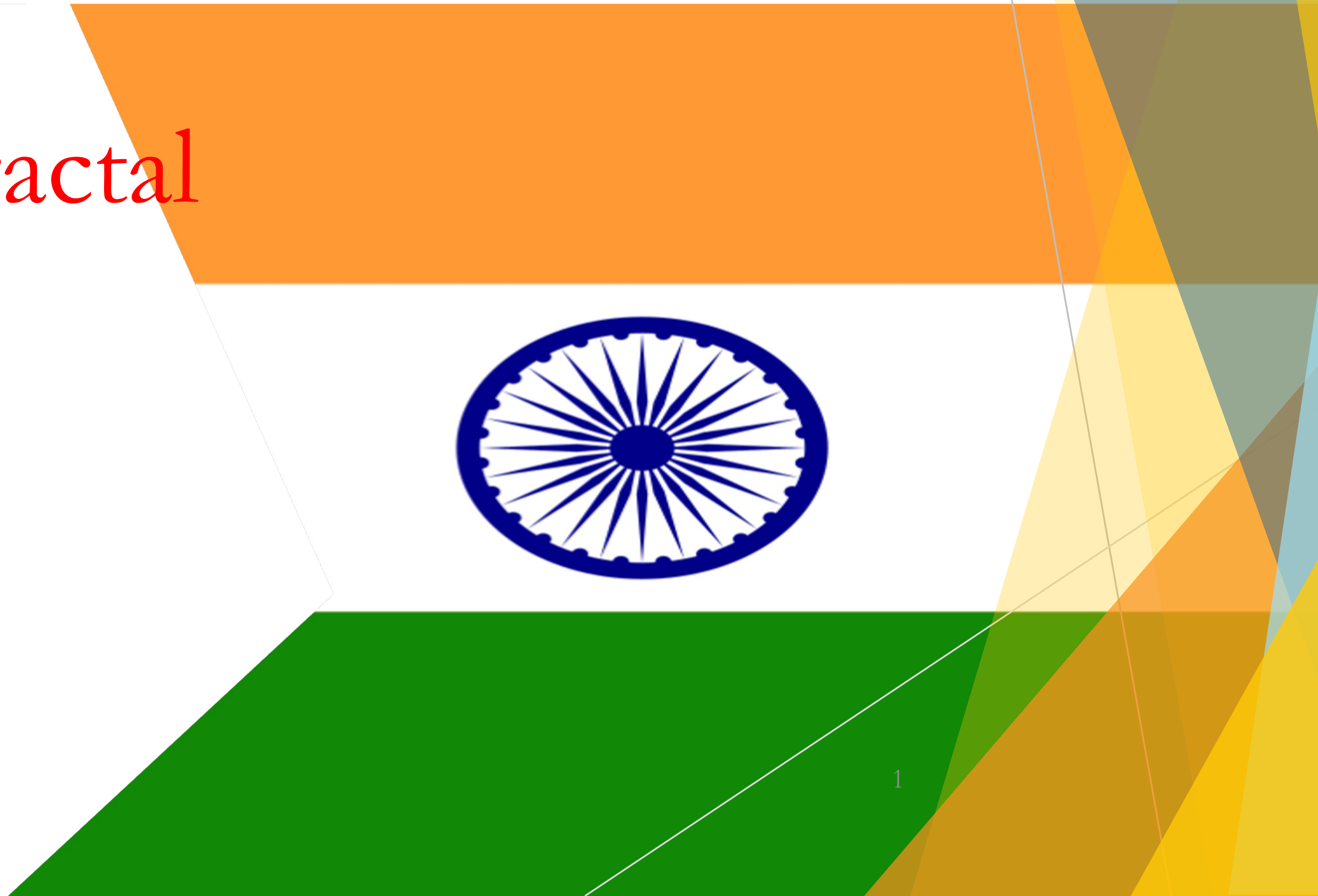
Problem 1 – Fractal Fingers

Shikhar Sehgal
India



Overview

A. Introduction

Problem Statement

The effect of fractal fingering can be observed if a droplet of an ink-alcohol mixture is deposited onto diluted acrylic paint. How are the **geometry and dynamics of the fingers** influenced by relevant parameters?

Defining the Problem Statement

- Fractal: A shape that contains multiple 'self-similar' structures that can be scaled. This leads to consideration of geometry in fractal dimensions, which (in fractal fingers) exceeds the Lebesgue covering dimension.
- Fractal Fingers: Fractal fingers, formed in fluids (in this case the ink-alcohol mixture) are a product of Saffman-Taylor instability (or viscous fingering). They exist as fractals, as previously defined.

Defining the Problem Statement + other notable terms

- Saffman-Taylor instability: This is the phenomenon when a less viscous fluid displaces a more viscous fluid in a mixture, leading to the formation of complex formations at the meeting point of both spatial regions.

Geometry and Dynamics:

- Fractal Dimensions: A ratio that compares a fractal pattern with the change in scale.
- Lebesgue covering dimensions: Euclidean dimensions (points = 0, lines = 1, 2-dimensional planes = 2, etc.)

Aims

- Controlled, reliable experimental setup for easy measurement and reproducible results.
- Quantitative theoretical model
- Relate theoretical model with experimental findings
- Note: all experimentations have uncertainty values of +-n% for n=least count as digital instruments were used (manual would be +-n/2)

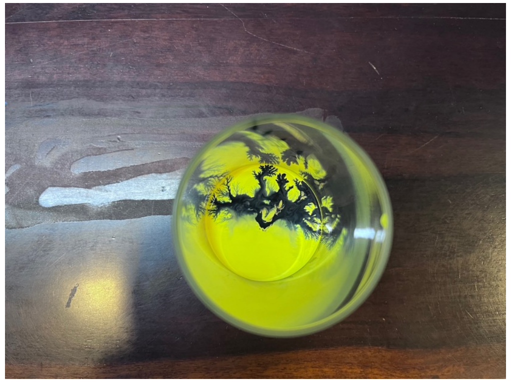
B. Quantitative and Qualitative Analysis

Qualitative Analysis of Saffman-Taylor Instability

- Saffman-Taylor instability refers to the creation of patterns in an unstable porous interface between two fluids (multiphase flow)
- The phenomenon occurs when a less viscous fluid displaces a more viscous fluid.
- One such example is when ink-ethanol of lower viscosity displaces acrylic paint in a petri dish, forming fractal fingers

Quantitative Analysis: Hausdorff Dimension

- The Hausdorff Dimension ~ $\dfrac{\ln(\text{number of fingers per formation})}{\ln(\text{number of major formations that make up the full structure})}$ or $\dfrac{\ln(\text{number of self}-\text{similar objects})}{\ln(\text{scale factor})}$
- This gives $\dfrac{\ln 3}{\ln 2} \approx 1.6$
- Surprisingly, this is the same as the Koch Snowflake fractal



Quantitative Analysis (Dynamics): Thin-film Equation

- We use the thin-film equation to estimate the top-to-bottom thickness of viscous fingers formed, forming the mathematical model for the fingers

$$\frac{\partial h}{\partial t} = -\nabla \cdot \mathbf{Q}$$

Differential of height with respect to time = fluid flux*negative delta operator

Lubrication Theory (Dynamics)

- When thickness is taken to be to negligible, we form a mathematical model for viscous fingers represented by the following:
- $\frac{dp}{dx} = \mu \times \frac{d^2u}{dz^2}$
- “Differential of fluid pressure with respect to x direction = fluid viscosity*double differential of fluid velocity with respect to z direction”

Darcy’s Law

- “Fluid flux = pressure difference*(permeability*Area)/(dynamic viscosity*distance)”
- Due to permeability not being a related factor, the accurate model is Ergun’s Equation (referenced later)

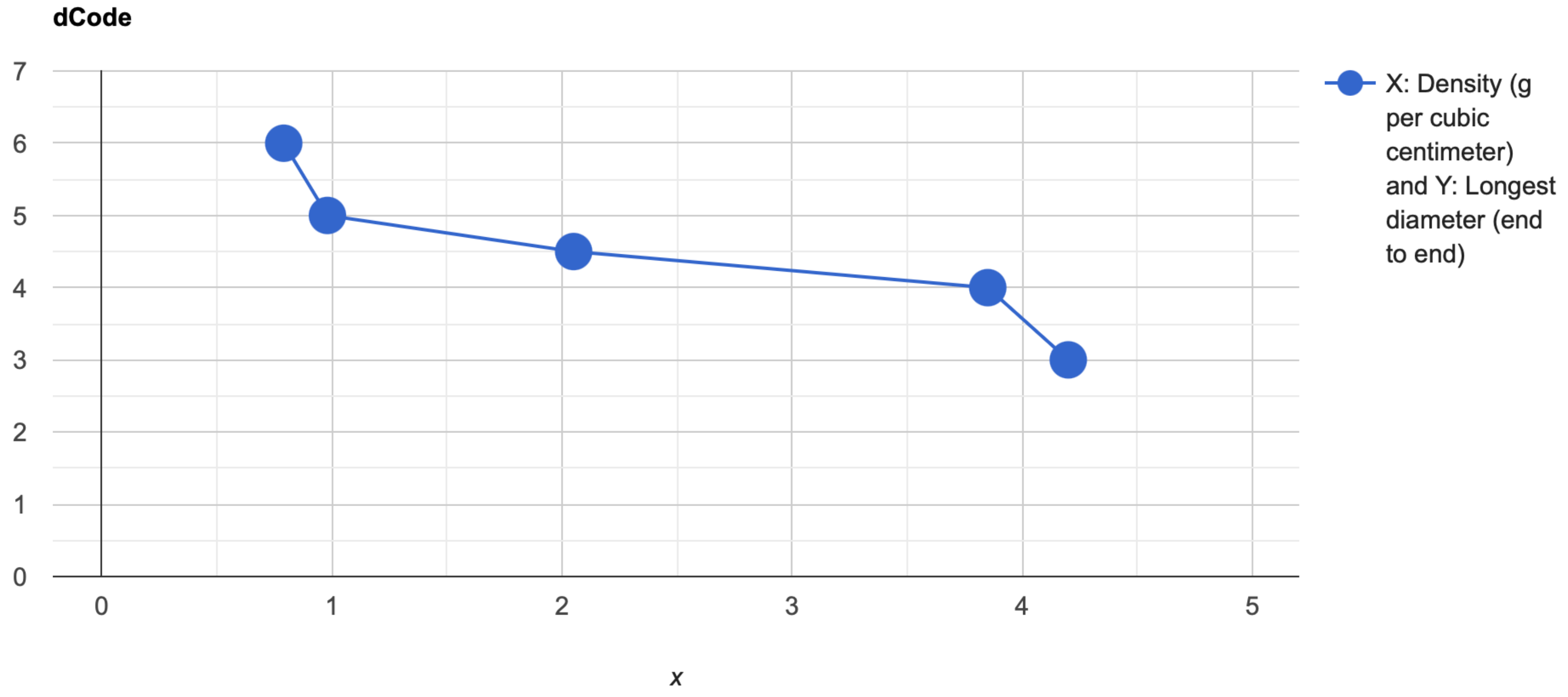
$$Q = \frac{kA}{\mu L}\,\Delta p$$

Comparison to The Mandelbrot Set (Boundary Conditions – Dynamics)

- The boundary of the Mandelbrot Set, though it has a Hausdorff dimension of 2, is similar to the parameterization of the fractal boundary of a fractal finger purely for drawing a comparison.

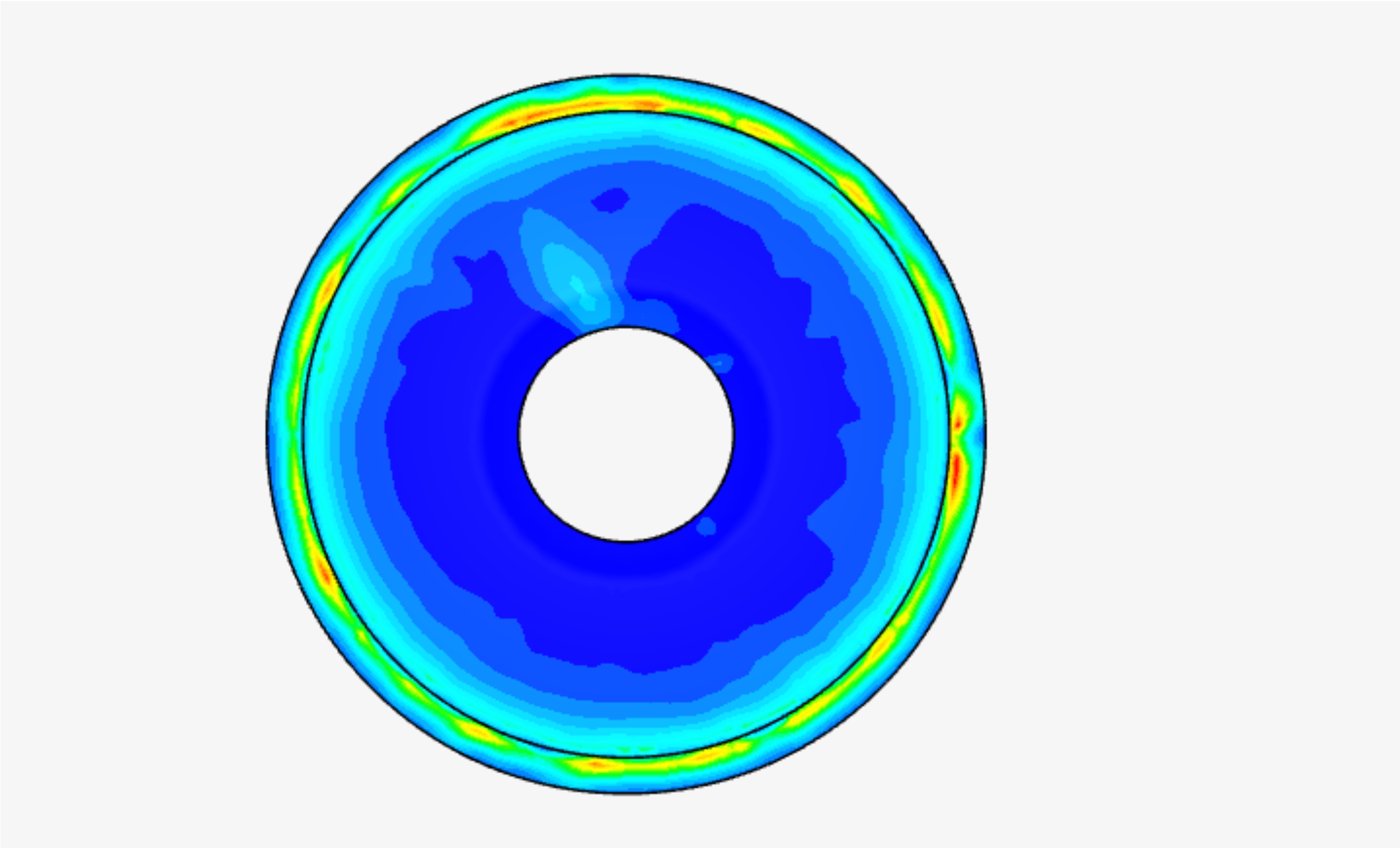
General diameter of spread

- Ink Density constant (0.6 gcm^{-3})



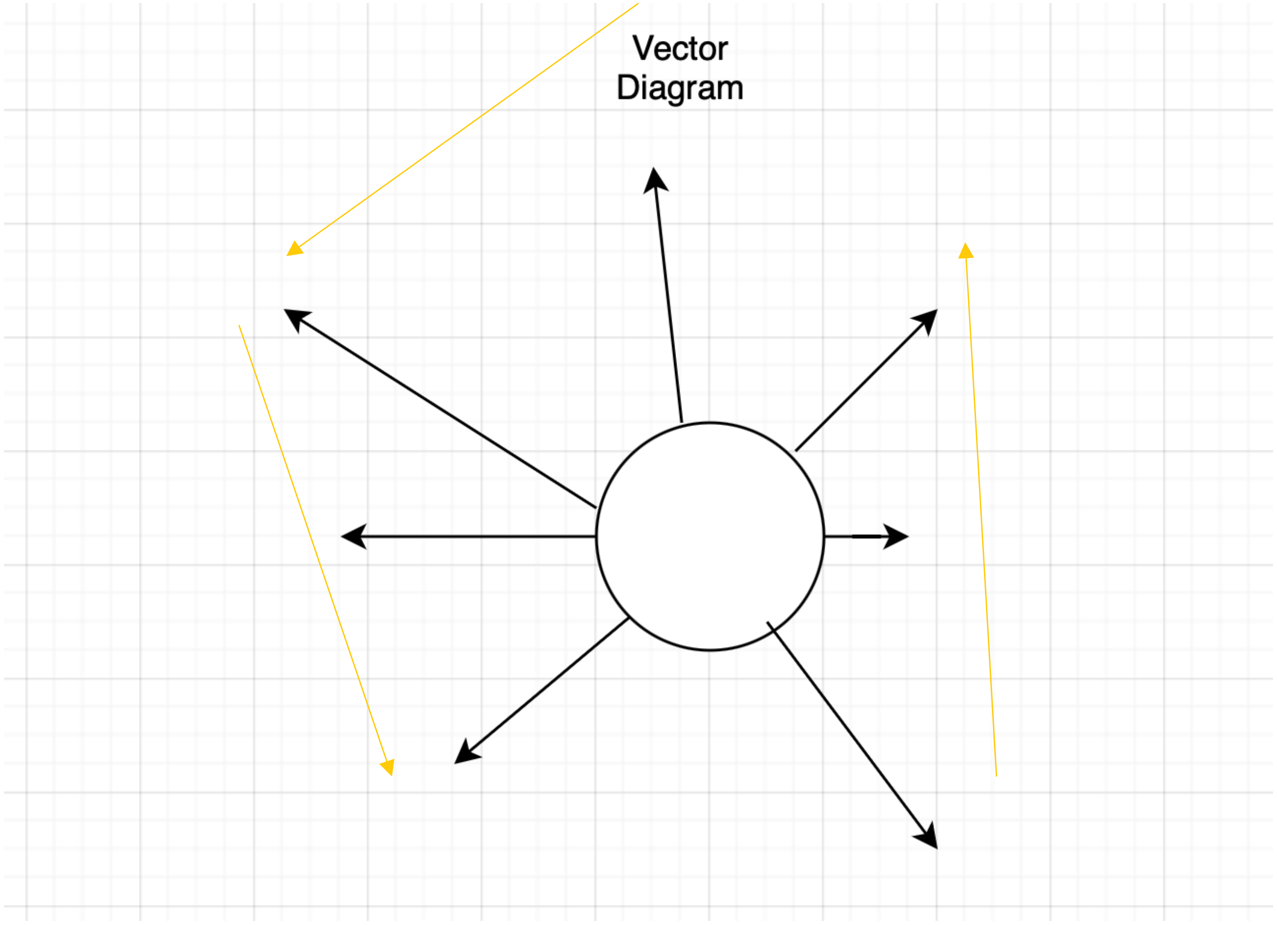
C. Simulations

Simulation A

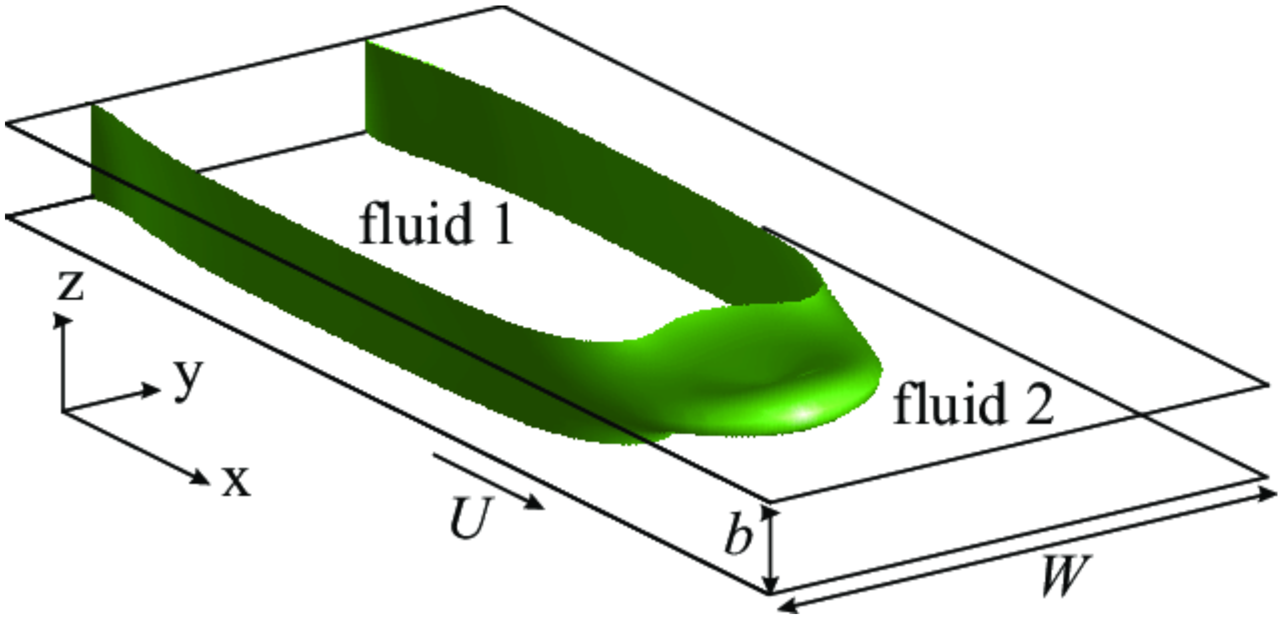


Simulation B (Vector Diagram)

- Tangents represent Euler Force

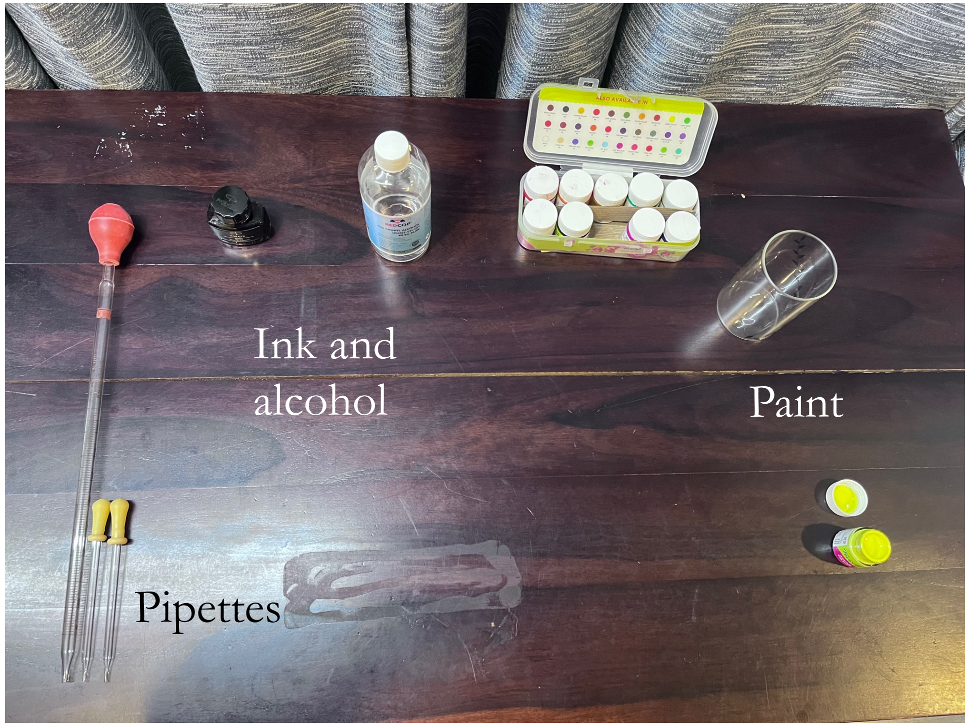


Simulation C (Ideal Condition of Boundary)



D. Experimental Setup

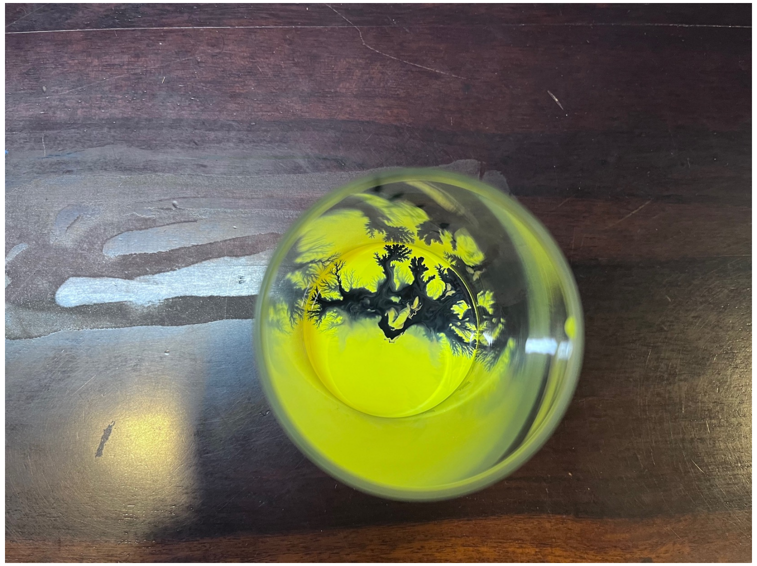
Experimental Set Up A. General Set-Up



Experimental Set Up B. Pipette/Dropper, Solute and Alternative Set-Ups


BROMINE
WATER
1
AMMONIUM
HYDROXIDE

Experimental Set Up C. Final Result



E. Factor Summary and Analysis

Property: State (Newtonian/Non-Newtonian)

- Newtonian fluids are fluids with a constant viscosity throughout their volume.
- Non-Newtonian fluids are fluids with a non-constant viscosity throughout their volume.
- The mixture of ink, alcohol and paint is a non-Newtonian fluid with time-dependent viscosity, i.e. viscosity changes with time.
- The sub-type of acrylic paint is viscoelastic.

Property: Viscoelasticity

- Viscoelasticity is the property by which a fluid may be both elastic and viscous by quality (e.g. Bingham plastic).
- Such a fluid (higher viscoelasticity) produces a narrow, tortous finger and branched pattern due to Saffman-Taylor instability

Direct Factors for Dynamics (Preliminary)

- The following theoretical equation represents direct factors affecting the growth rate of viscous fingers (dynamics) in Saffman-Taylor instability.

$$\sigma = \frac{kV\left(\mu_2 - \mu_1\right) - \gamma H_f k^3}{\mu_1 + \mu_2},$$

Causative Factors (Geometry and Dynamics)

- Observation time
- Viscosity
- Density
- Surface Tension Gradient
- Adverse Pressure Gradient
- Volume of Solute and solvent
- Miscibility
- Reynold’s Number

Atmospheric Pressure

Euler Force (simply note its existence)

Peclet Number

Effective Permeability

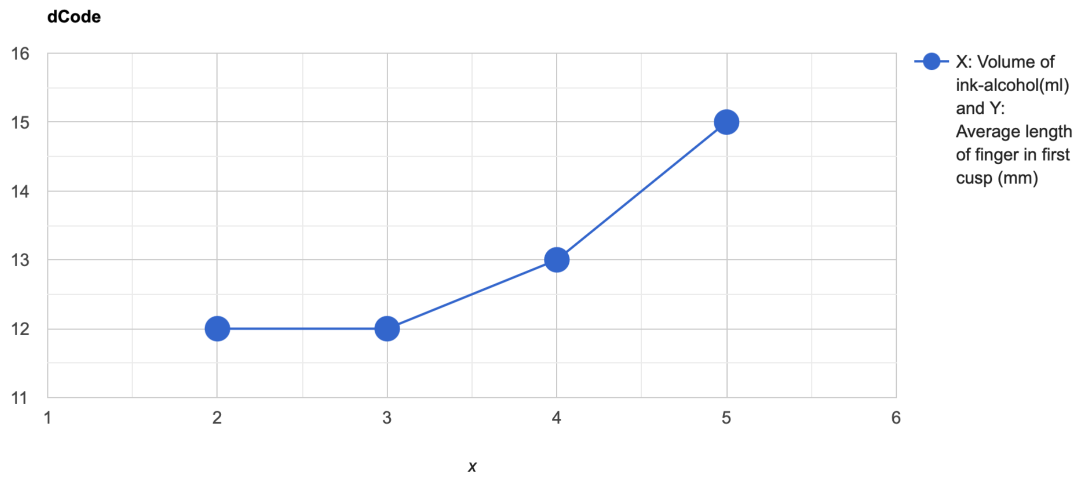
Height of drop

Observation Time

- For shorter Observation Time (<~37.7 minutes) in experiments leads to longer, thinner fractal fingers.
- Longer Observation Time (>~37.7) leads to partial dilution of the fractal fingers for 2ml solute, with only fewer shorter and more circular fingers visible. (example below – ~1 hour 27 min taken for full dissolution)

Volume of solute or solvent

Solute (in acrylic paint), Solvent – 25ml



Solvent (with ink-alcohol), Solute – 2ml

- No Noticeable changes, max 1mm but that is likely just random error. More area for solvent, i.e larger petri dish leads to less restrictions on max diameter.

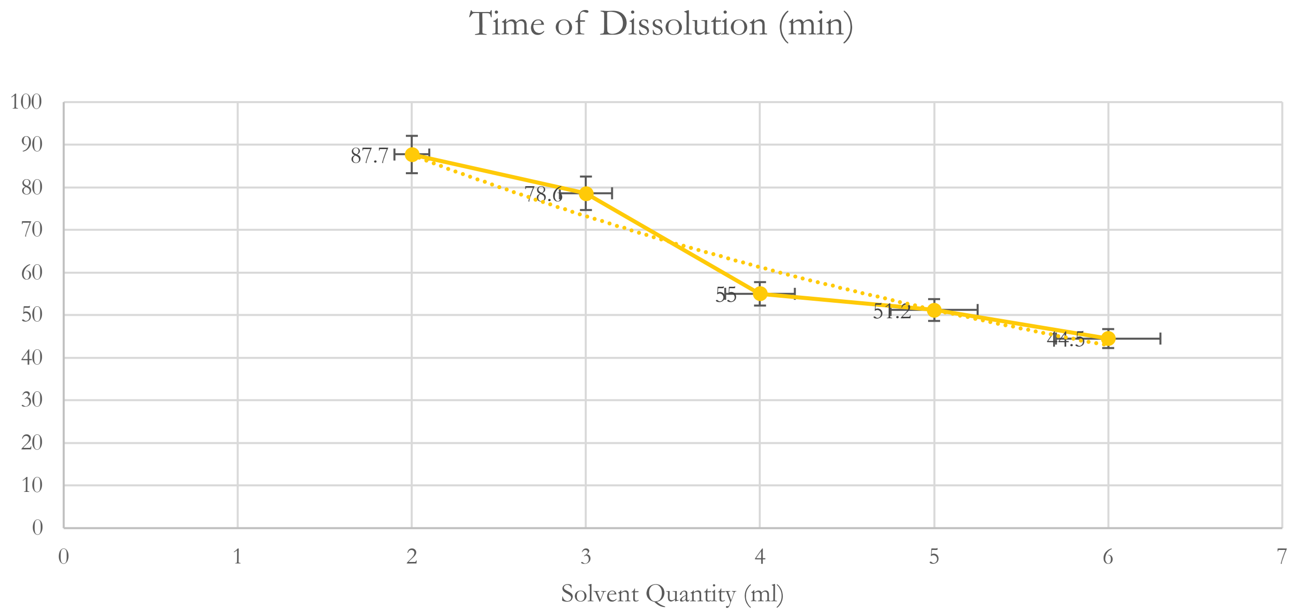
Correlation of thickness of fingers, volume of solute and observation time through the Thin-Film Equation

- For: $\frac{\partial h}{\partial t} = -\nabla \cdot \mathbf{Q}$

- The more fluid added, the thicker (from top to base) the formation of fractal fingers is. Counterintuitively, since $\frac{\partial h}{\partial t}$ is negative-sloped, the time taken for dissolution due to more volume, cohesive forces, and gravity, is reduced.

Experimental Verification

- Time of dissolution: Formation visible but faded to below the surface of acrylic paint*



Viscosity (not accounting for zero-viscosity superfluids)

- Viscosity accounts for the fluid's resistance to flow (or fluid friction).
- For a higher viscosity, the flow of the fluid being added is lower and travels a lower distance (in accordance with Darcy's Law).
- Darcy's Law: instantaneous flow rate = - permeability*pressure drop/dynamic viscosity (explained in quantitative analysis)
- This leads to formation of shorter and fewer fractal fingers.
- Rayleigh-Taylor instability for more viscous fluid displacing less viscous
- Thickness directly proportional to viscosity (Ergun's equation)

Dynamic Viscosity of Solvent

X: distance from center in cm, y: viscosity in Pa s in viscometer

	abscissa x	ordinate y or f(x)
1	0	0.098
2	2	0.092
3	4	0.086
4	6	0.082
5	8	0.078
6	...	...
7	...	...

X: viscosity, y: thickness in mm

	abscissa x	ordinate y or f(x)
1	0.098	N/A
2	0.092	0.53
3	0.086	0.32
4	0.082	0.26
5	0.078	0.17
6	...	...
7	...	...

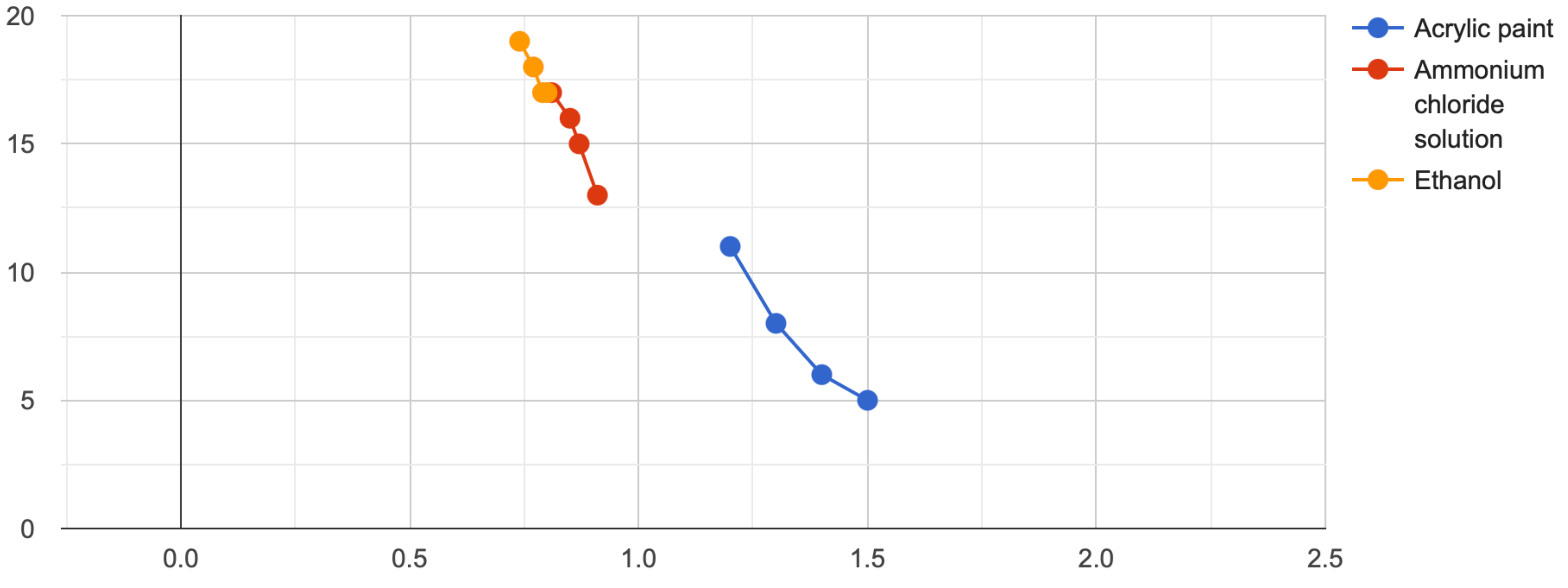
No literature exists for verification*

Density

- While density can be assumed to be constant for a theoretical experiment, it is notable that density decreases as one moves outward from the centre of the point of contact between the ink drops and the solvent.
- Reasoning: With the decrease in pressure gradient towards the outer parts of the mixture, the volume decreases and hence density decreases.
- Shorter and fewer fractal fingers are hence formed.

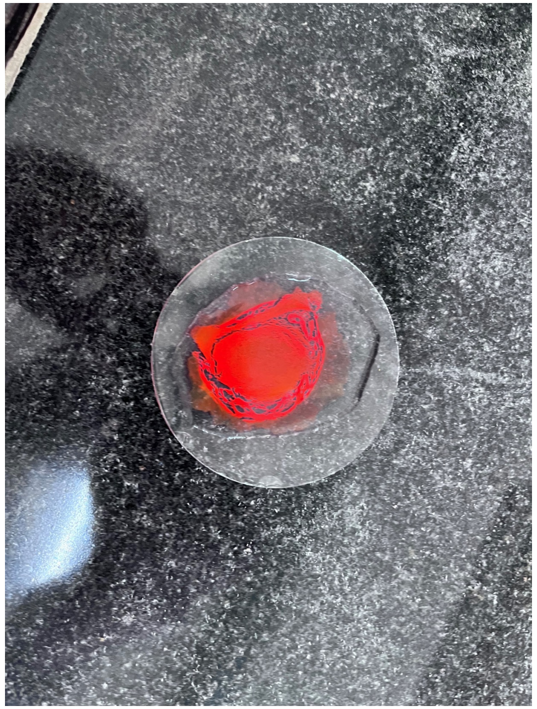
Graphing finger length for different densities of solvents

- Y-axis: F(length) in mm (+-0.05), X-axis: density in gcm^{-3} (+-0.02) Control factor q of solute



Experimental Verification

Density = 0.8 gcm^{-3}



Density = 0.67 gcm^{-3}

Adverse Pressure Gradient

- The pressure of the fluid is highest at the centre of the beaker, where ink is dropped.
- The pressure of fluid decreases outward in the fluid mixture.
- Thickness of fractal fingers decreases with outward movement, hence showing a positive relationship.
- However, in accordance with Darcy's law, there is an increase in volumetric flow rate (due to positive proportionality with decrease in pressure) – hence fractal fingers will appear longer

Surface Tension Gradient

- According to the Young-Laplace equation, change in pressure is directly proportional to pressure difference. Hence, this parameter follows the same rule as the Adverse Pressure Gradient.

$$\Delta P = (P_{\text{int}} - P_{\text{ext}}) = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

- This can be equated to the Lubrication equation to formulate: viscosity difference*double differential of velocity with respect to perpendicular axis + surface tension*mean curvature = 0

Dynamic Surface Tension of Solvent

X: distance from center in cm, y: surface tension in mNm^{-1} in tensiometer

	abscissa x	ordinate y or f(x)
1	0	22.3
2	2	22.1
3	4	20.9
4	6	20.4
5	8	19.9
6	...	...
7	...	...

Readings here (average of $(21.1).10^{-4}$ match *Shan et al.* average value of $(21.4).10^{-4}$ at RTP on a LBM simulation (difference is systemic error due to Mumbai climate)

X: surface tension, y: thickness in mm

	abscissa x	ordinate y or f(x)
1	22.3	N/A
2	22.1	0.51
3	20.9	0.34
4	20.4	0.29
5	19.9	0.18

The values yielded here help verify the general accuracy of viscosity

Reynold's Number

- Reynold's number is the ratio of inertial forces to viscous forces in a fluid. It may be represented as:
- Re = $\frac{pVD}{u}$

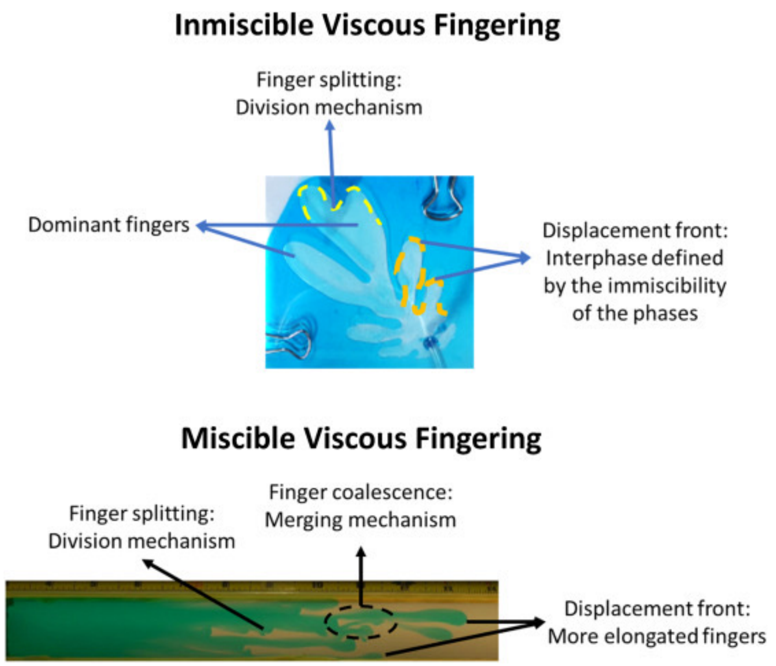
For Re=Reynolds number,

p=fluid density, V = average fluid velocity, D = petri dish/bowl diameter, u = average fluid dynamic viscosity

- It is generally used to check for turbulent or laminar flow in a fluid. Due to a low Reynold's number for Stokes flow in this problem, there is theoretically laminar flow except in the case of obstructions, but experimentally turbulent flow is prevalent

Miscibility

- Affects symmetry, interface, and division mechanisms. Yields larger surface area for fingers for higher miscibility.
- **Here, miscibility cannot vary.**



Using the formulation to find velocity of fluid flow (dynamics)

- At a particular point: dynamic viscosity, surface tension, and mean curvature can all be calculated.
- Using the formulation: The Young-Laplace equation can be equated to the Lubrication equation to formulate:

Using the formulation to find velocity of fluid flow (dynamics)

- $(\mu_2\text{-}\mu_1)\times\frac{dv^2}{d^2z} + \frac{\gamma\times(\frac{1}{R1}+\frac{1}{R2})}{dx} = 0$

- Or:

viscosity difference*double differential of velocity with respect to perpendicular axis + surface tension*mean curvature = 0

- Can give the value for jerk of fluid flow, and hence acceleration (and then velocity), between two points.
- This can also be used to find fluid pressure between 2 points (which lacked instrumentation – differential manometer)

Alternative Formulation using different variables

- Equating Darcy's Law and the Young-Laplace Equation,

$$\gamma\times\left(\frac{1}{R1}+\frac{1}{R2}\right) = Q.\frac{\mu L}{kA}$$

Hence, we can calculate final fluid flux Q of the experiments using:

$$\frac{\{\gamma\times\left(\frac{1}{R1}+\frac{1}{R2}\right)\}.kA}{\mu L}$$

Mathematical Model to Summarise The Friction Factor/Viscosity

- Geometry (also Dynamics) is affected to largest extent by friction, which is explained by all previously mentioned factors. Following is the model to understand the factors affecting geometry of the fractal fingers. Packed bed becomes the Hele-Shaw cell.
- $f_p = 150/Gr_p + 1.75$
- Can be expressed as:

$$f_p = \frac{\Delta p}{L} \frac{D_p}{\rho v_s^2} \left(\frac{\epsilon^3}{1-\epsilon} \right) \text{ and } Gr_p = \frac{\rho v_s D_p}{(1-\epsilon)\mu} = \frac{Re}{(1-\epsilon)};$$

Ignored Factors

- **Effective Permeability** refers to the ability of the fluid (ink) to flow through the paint, relative to the paint. This is same for all ink used in experiment
- **Atmospheric Pressure** as a downward vector would cause a reduction in thickness but no change need be accounted for in experiment and $P_f = P_{at} + dgh$ is not applied to thin film.
- Since the ink-alcohol has not diffused through the solvent downward, **Peclet number** can be ignored.

F. Additional Elements

Stokes Flow

- In a solvent (paint) of higher viscosity, the advective inertial forces of the ink are lower than the viscous forces of paint.
- This leads to the formation of a Stokes flow.
- The fluid velocity (and hence spread of viscous fingers) is very slow in this case.
- The rate of Stokes flow may be represented using the Reynold's number.
- This may be represented using the Navier-Stokes Equation.

Hele-Shaw Flow

- Hele-Shaw Flow is the ideal Stokes flow for this experiment, within the thin parallel plates of a Hele-Shaw cell.
- However, this WILL NOT EXIST IN THE CASE OF THIS EXPERIMENT unless done in these special conditions as the problem statement does NOT REQUIRE THE HELE-SHAW CELL.

G. Conclusion

Conclusion

Experimentally verified theoretical expectations.

Factors affecting viscous flow were successfully found.

Special Conditions for theoretical applications and Limitations:

- Use of a Hele-Shaw Cell
- Stokes flow is the motion of a fluid where inertial force<viscous force.
- When this fluid flows between parallel flat plates with negligible gap, it is the condition of a Hele-Shaw cell.
- Note: These conditions are not used for this experiment, but are the ideal conditions for the experiment to take place.
- Some instrumentations are unavailable for research (for pressure gradient and miscibility)

Obstructions

- Obstructions cause distortions in the shape of the fractal fingers.
- Obstructions in the petri dish may cause complete absence of formation of fractal fingers, or smaller/larger fingers formed in the petri dish.
- They will cause turbulent flow of ink through paint.

H. Works Cited

Works Cited

- **Graphs made with dcode, Measurements made with laboratory equipment (CAJCS – Mumbai), Tracker, and CellProfiler. All original.**
- **Simulations made with SimScale and draw.io**
- Spontaneous fingering between miscible fluids – ScienceDirect
- How to Calculate Diffusion Rate (sciencing.com)
- https://stemfellowship.org/iypt-references/problem1/
- https://physlets.org/tracker/trackerJS/
- https://cellprofiler.org/releases
- (PDF) Numerical simulation of the Saffman-Taylor instability in the Hele-Show cell (researchgate.net)

Works Cited

[Schematic of the apparatus [33] used to measure pull-off forces.... | Download Scientific Diagram (researchgate.net)]

[Viscous Fingering in Porous Media | Annual Review of Fluid Mechanics (annualreviews.org)]

[fluids-04-00053-v3.pdf]

https://www.researchgate.net/figure/Viscous-fingers-at-the-air-nematic-interface-d-320-m-p-e-5-mbar-a-E-off-t_fig3_237640338

https://www.researchgate.net/figure/a-The-shape-of-the-interface-in-the-xz-plane-for-a-viscosity-contrast-r-04-as-a_fig2_256488037

Works Cited

https://www.sciencedirect.com/science/article/pii/S2405844021017175

https://www.researchgate.net/figure/Velocity-profiles-for-channel-flow-Newtonian-n1-non-Newtonian-shear-thinning_fig3_254532628

https://www.researchgate.net/figure/Illustration-of-lattice-Boltzmann-LBGK-simulation-of-viscous-fingering-flow-direction_fig3_231146403

https://www.sciencedirect.com/science/article/abs/pii/S0045793009001984

https://www.semanticscholar.org/paper/VISCOUS-FINGERING-IN-MISCIBLE%2C-IMMISCIBLE-AND-Grosfils-Boon/00bb86b5fc086b3cbee0ca84a68402ead18a159f/figure/1

Works Cited

https://www.sciencedirect.com/science/article/abs/pii/S0045793009001984

https://cdnsciencepub.com/doi/10.1139/cjp-2019-0120

https://www.sciencedirect.com/science/article/abs/pii/S0920410521016211

https://link.springer.com/article/10.1007/s11242-021-01629-8

https://hrcak.srce.hr/file/290558

https://cdnsciencepub.com/doi/10.1139/cjp-2019-0120

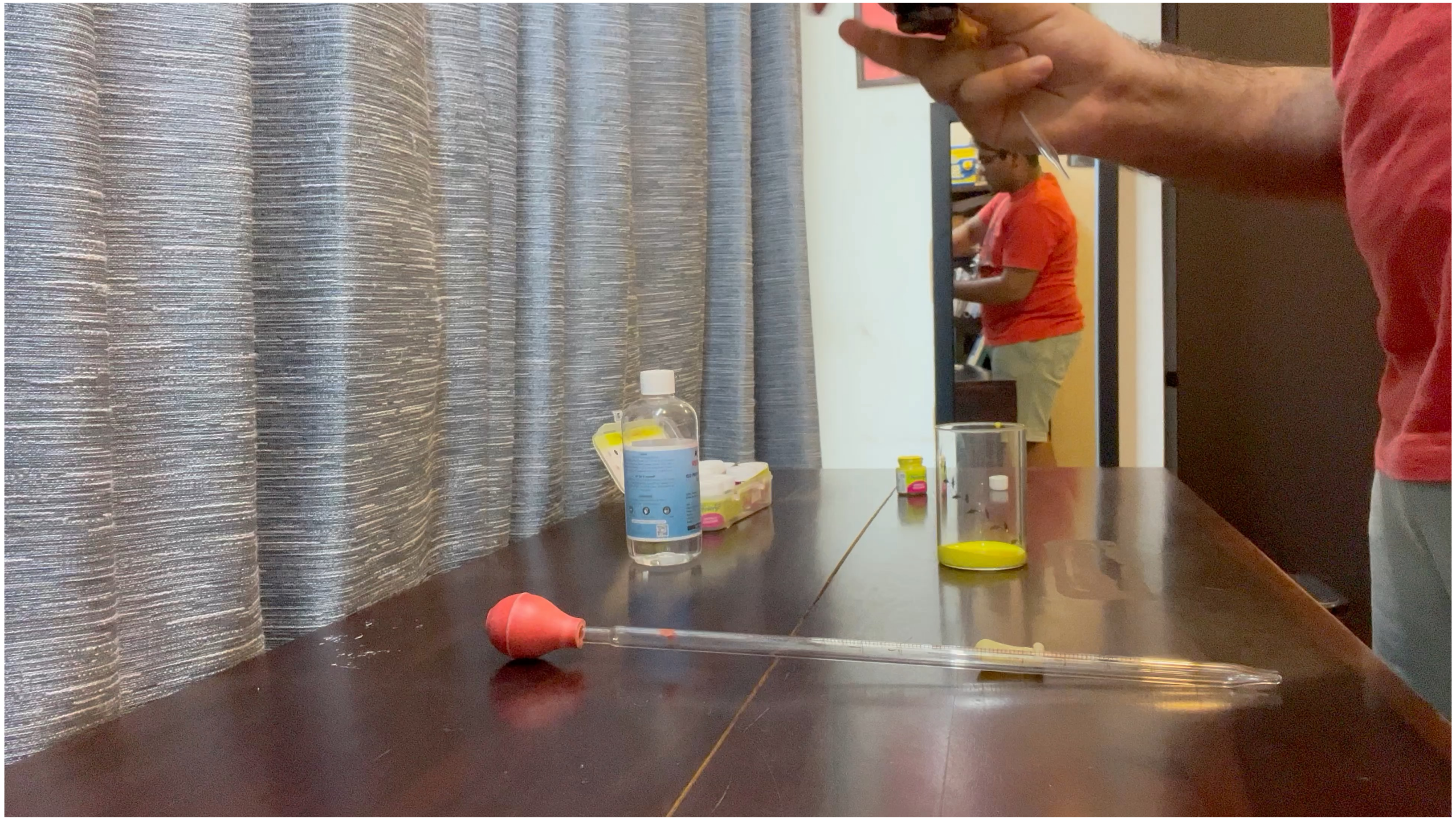
https://aip.scitation.org/doi/10.1063/5.0045051

Thank you.

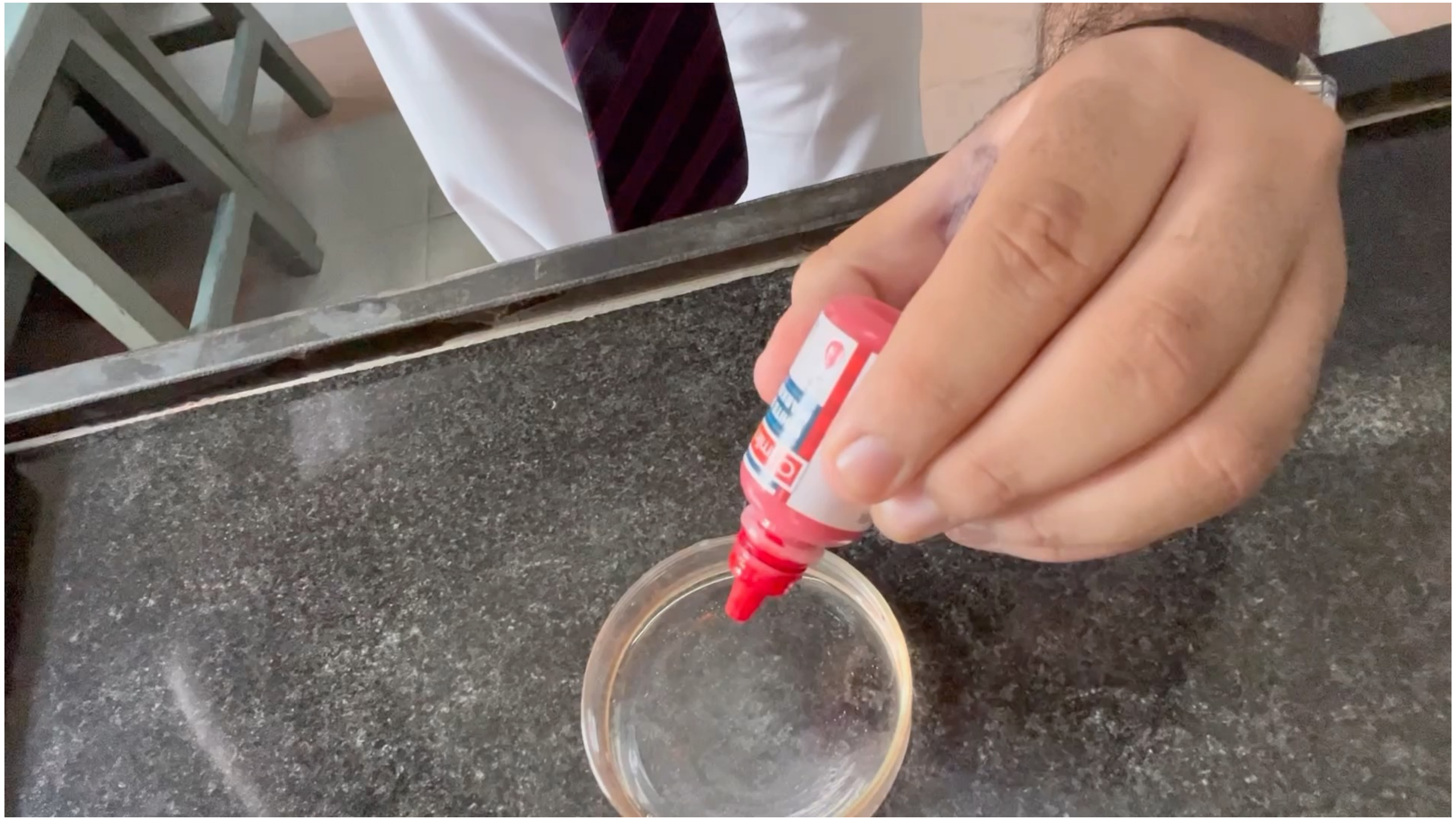
- Shikhar Sehgal Team India

A. Video Demonstrations

Video Demonstration A



Video Demonstration B



Video Demonstration C (Slow Motion)