

An Algebra Problem from the University of Waterloo's 2023 Euclid Exam

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Introduction

In this article, I apply algebra properties and elementary factorisation techniques to solve a problem from the University of Waterloo's 2023 Euclid Exam. The problem is particularly interesting as it initially seems deceptive to most high schoolers (this is the larger independent subpart of the question, but in total the world average for this problem was 3.3/10 according to the results booklet released by the University); however, elementary techniques yield a simple solution.

Problem

"Determine the number of quadruples of positive integers (a, b, c, d) with $a < b < c < d$ that satisfies both of the following systems of equations:

$$ac + ad + bc + bd = 2023 \dots\dots\dots(1)$$

$$a + b + c + d = 296 \dots\dots\dots(2)$$

Key points

First, factorise equation (1) correctly. Take the factors as two distinct variables, instead of 4. Do the same for equation (2) without factorisation. Factorise 2023, then check for all combinations – remembering that $a < b < c < d$ (Remember to count correctly).

Full Solution

Factorising equation 1,

$$ac + ad + bc + bd$$

$$= a(c + d) + b(c + d)$$

$$= (a + b)(c + d)$$

$$\text{Equation 2: } a + b + c + d = 296$$

$$\text{Let } (a+b) = x,$$

$$\text{Let } (c+d) = y,$$

$$\text{Hence, } xy = 2023 \text{ and } x+y = 296.$$

$$\text{Since } a, b, c, d \in \mathbb{Z},$$

x and y are also integers;

$$\text{Factorise } xy = 2023: 1 \text{ and } 2023, 7 \text{ and } 289, 17 \text{ and } 119,$$

$$\text{But } x+y = 296, \text{ so } x = 7 \text{ and } y = 289$$

$$\text{Since } a < b \text{ and } x = a+b,$$

$$x \text{ can be made up of the elements of the following sets: } \{a, b\} = \{1, 6\}; \{2, 5\}; \{3, 4\}$$

Now with these 3 possibilities for x , there are multiple possibilities for $y = 289$ such that $y = c+d$ and $c < d$;

This is simple and does not require combinatorics of any sort.

For $\{a, b\} = \{1, 6\}$,

$\{c, d\} = \{7, 282\}; \{8, 281\} \dots \{144, 145\}$ (Remember that $c > 6$ as $c > b$)

There are 138 such combinations.

For $\{a, b\} = \{2, 5\}$,

$\{c, d\} =$ All of those mentioned above, as well as $\{6, 283\}$ since $c > b$ means $c > 5$ this time.

There are $(138+1 = 139)$ such combinations.

Repeat this one final time for $\{a, b\} = \{3, 4\}$ and there are 140 combinations.

Result

Number of quadruples $\{a, b, c, d\}$ that satisfy equations (1) and (2) under the restraint $a < b < c < d = 138 + 139 + 140 = \mathbf{417}$.